1 Poisson regression with two random effects

The data consist of counts of the number of epileptic attacks in 4 periods on 59 subjects, y_{it}, $i = 1, \ldots, 59$, $t = 1, \ldots, 4$.
We want to adjust for 4 subject-specific covariates x_{ij}, $j = 1, \ldots, 4$, namely the baseline, treatment, baseline by treatment interaction and age. There is also a single time varying covariate x_{it1} that denotes whether or not this is visit 4.

The model says that y_{it} is Poisson with mean μ_{it}. So that,

$$p(y_{it}) = \frac{exp(-\mu_{it})\mu_{it}^{y_{it}}}{y_{it}!}$$

and the linear predictor is given by,

$$log(\mu_{it}) = \eta_{it} = \beta_0 + \sum_{j=1}^{4} \beta_j x_{ij} + \beta_5 x_{it1} + u_i + e_{it}$$

where u_i is a subject specific random effect and e_{it} is a subject by period random effect.

We will assume that the random effects are both normally distributed.

$$u_i \sim N(0, \tau_u) \qquad e_{it} \sim N(0, \tau_e)$$

where τ_u and τ_e are precisions.

The priors for the model will take the form of normal distribution for the regression coefficients, β, and gamma distributions for the precisions, τ.

$$\beta_j \sim N(m_j, t_j) \; j = 0, \ldots, 5 \qquad \tau_j \sim G(a_j, b_j) \; j = u, e$$

If $\theta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \tau_u, \tau_e)$ we can write $p(y|\theta, u, e)$ for the likelihood of the data, $p(u|\theta)$ and $p(e|\theta)$ for the random effect distibutions and $\pi(\theta)$ for the prior. The joint posterior will be proportional to,

$$\left[\prod_{i=1}^{59}\prod_{t=1}^{4} p(y_{it}|\theta, u_i, e_{it})\right]\left[\prod_{i=1}^{59} p(u_i|\theta)\right]\left[\prod_{i=1}^{59}\prod_{t=1}^{4} p(e_{it}|\theta)\right]\pi(\theta)$$

We need to calculate the log-posterior which will have the form,

$$\left[\sum_{i=1}^{59}\sum_{t=1}^{4} log(p(y_{it}|\theta, u_i, e_{it}))\right] + \left[\sum_{i=1}^{59} log(p(u_i|\theta))\right] + \left[\sum_{i=1}^{59}\sum_{t=1}^{4} log(p(e_{it}|\theta))\right] + log(\pi(\theta))$$

In fact we only need the log-posterior to within a constant so we can leave off additive terms that do not depend on the parameters. This gives.

$$log(p(y_{it}|\theta, u_i, e_{it})) = -\mu_{it} + y_{it} log(\mu_{it}) = -exp(\eta_{it}) + y_{it}\eta_{it}$$

$$log(p(u_i|\theta)) = \frac{1}{2}log(\tau_u) - \frac{1}{2}\tau_u u_i^2$$

$$log(p(e_{it}|\theta)) = \frac{1}{2}log(\tau_e) - \frac{1}{2}\tau_e e_{it}^2$$

$$log(\pi(\theta)) = -\frac{1}{2}\sum_{j=0}^{5} t_j(\beta_j - m_j)^2 + (a_u - 1)log(\tau_u) - \frac{\tau_u}{b_u} + (a_e - 1)log(\tau_e) - \frac{\tau_e}{b_e}$$

When we update the a particular parameter we only need to include the terms from the log-posterior that include or depend on that parameter. So for β_j we need,

$$\left[\sum_{i=1}^{59}\sum_{t=1}^{4} log(p(y_{it}|\theta, u_i, e_{it}))\right] + log(\pi(\beta_j))$$

or

$$\left[\sum_{i=1}^{59}\sum_{t=1}^{4} -\mu_{it} + y_{it}log(\mu_{it})\right] - \frac{1}{2}t_j(\beta_j - m_j)^2)$$

or

$$\left[\sum_{i=1}^{59}\sum_{t=1}^{4} -exp(\eta_{it}) + y_{it}\eta_{it}\right] - \frac{1}{2}t_j(\beta_j - m_j)^2)$$

For τ_u we need,

$$\left[\sum_{i=1}^{59} \frac{1}{2}log(\tau_u) - \frac{1}{2}\tau_u u_i^2\right] + (a_u - 1)log(\tau_u) - \frac{\tau_u}{b_u}$$

For τ_e we need,

$$\left[\sum_{i=1}^{59}\sum_{t=1}^{4} \frac{1}{2}log(\tau_e) - \frac{1}{2}\tau_e e_{it}^2\right] + (a_e - 1)log(\tau_e) - \frac{\tau_e}{b_e}$$

To update the random effect u_i we need,

$$\left[\sum_{t=1}^{4} -\mu_{it} + y_{it}log(\mu_{it})\right] - \frac{1}{2}\tau_u u_i^2$$

To update the random effect e_{it} we need,

$$[-\mu_{it} + y_{it}log(\mu_{it})] - \frac{1}{2}\tau_e e_{it}^2$$

2 Gibbs sampling

The form of the conditional log-posterior for τ_u is,

$$\left[\sum_{i=1}^{59} \frac{1}{2}log(\tau_u) - \frac{1}{2}\tau_u u_i^2\right] + (a_u - 1)log(\tau_u) - \frac{\tau_u}{b_u}$$

if we collect together similar terms we get,

$$(a_u - 1 + \frac{59}{2})log(\tau_u) - \tau_u \left[\frac{1}{b_u} + \frac{1}{2}\sum_{i=1}^{59} u_i^2\right]$$

By comparison with the log-prior for the gamma distribution this can be recognised as the the form of another Gamma distribution this time with parameters

$$G\left(a_u + 29.5, \left[\frac{1}{b_u} + \frac{1}{2}\sum_{i=1}^{59} u_i^2\right]^{-1}\right)$$

Similar calculations for τ_e lead to,

$$G\left(a_e + 118, \left[\frac{1}{b_e} + \frac{1}{2}\sum_{i=1}^{59}\sum_{t=1}^{4} e_{it}^2\right]^{-1}\right)$$