

1 Poisson regression with two random effects

The data consist of counts of the number of epileptic attacks in 4 periods on 59 subjects, y_{it} , $i = 1, \dots, 59$, $t = 1, \dots, 4$.

We want to adjust for 4 subject-specific covariates x_{ij} , $j = 1, \dots, 4$, namely the baseline, treatment, baseline by treatment interaction and age. There is also a single time varying covariate x_{it1} that denotes whether or not this is visit 4.

The model says that y_{it} is Poisson with mean μ_{it} . So that,

$$p(y_{it}) = \frac{\exp(-\mu_{it})\mu_{it}^{y_{it}}}{y_{it}!}$$

and the linear predictor is given by,

$$\log(\mu_{it}) = \eta_{it} = \beta_0 + \sum_{j=1}^4 \beta_j x_{ij} + \beta_5 x_{it1} + u_i + e_{it}$$

where u_i is a subject specific random effect and e_{it} is a subject by period random effect.

We will assume that the random effects are both normally distributed.

$$u_i \sim N(0, \tau_u) \quad e_{it} \sim N(0, \tau_e)$$

where τ_u and τ_e are precisions.

The priors for the model will take the form of normal distribution for the regression coefficients, β , and gamma distributions for the precisions, τ .

$$\beta_j \sim N(m_j, t_j) \quad j = 0, \dots, 5 \quad \tau_j \sim G(a_j, b_j) \quad j = u, e$$

If $\theta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \tau_u, \tau_e)$ we can write $p(y|\theta, u, e)$ for the likelihood of the data, $p(u|\theta)$ and $p(e|\theta)$ for the random effect distributions and $\pi(\theta)$ for the prior. The joint posterior will be proportional to,

$$\left[\prod_{i=1}^{59} \prod_{t=1}^4 p(y_{it}|\theta, u_i, e_{it}) \right] \left[\prod_{i=1}^{59} p(u_i|\theta) \right] \left[\prod_{i=1}^{59} \prod_{t=1}^4 p(e_{it}|\theta) \right] \pi(\theta)$$

We need to calculate the log-posterior which will have the form,

$$\left[\sum_{i=1}^{59} \sum_{t=1}^4 \log(p(y_{it}|\theta, u_i, e_{it})) \right] + \left[\sum_{i=1}^{59} \log(p(u_i|\theta)) \right] + \left[\sum_{i=1}^{59} \sum_{t=1}^4 \log(p(e_{it}|\theta)) \right] + \log(\pi(\theta))$$

In fact we only need the log-posterior to within a constant so we can leave off additive terms that do not depend on the parameters. This gives.

$$\log(p(y_{it}|\theta, u_i, e_{it})) = -\mu_{it} + y_{it} \log(\mu_{it}) = -\exp(\eta_{it}) + y_{it} \eta_{it}$$

$$\log(p(u_i|\theta)) = \frac{1}{2}\log(\tau_u) - \frac{1}{2}\tau_u u_i^2$$

$$\log(p(e_{it}|\theta)) = \frac{1}{2}\log(\tau_e) - \frac{1}{2}\tau_e e_{it}^2$$

$$\log(\pi(\theta)) = -\frac{1}{2} \sum_{j=0}^5 t_j (\beta_j - m_j)^2 + (a_u - 1)\log(\tau_u) - \frac{\tau_u}{b_u} + (a_e - 1)\log(\tau_e) - \frac{\tau_e}{b_e}$$

When we update the a particular parameter we only need to include the terms from the log-posterior that include or depend on that parameter. So for β_j we need,

$$\left[\sum_{i=1}^{59} \sum_{t=1}^4 \log(p(y_{it}|\theta, u_i, e_{it})) \right] + \log(\pi(\beta_j))$$

or

$$\left[\sum_{i=1}^{59} \sum_{t=1}^4 -\mu_{it} + y_{it} \log(\mu_{it}) \right] - \frac{1}{2} t_j (\beta_j - m_j)^2$$

or

$$\left[\sum_{i=1}^{59} \sum_{t=1}^4 -\exp(\eta_{it}) + y_{it} \eta_{it} \right] - \frac{1}{2} t_j (\beta_j - m_j)^2$$

For τ_u we need,

$$\left[\sum_{i=1}^{59} \frac{1}{2} \log(\tau_u) - \frac{1}{2} \tau_u u_i^2 \right] + (a_u - 1)\log(\tau_u) - \frac{\tau_u}{b_u}$$

For τ_e we need,

$$\left[\sum_{i=1}^{59} \sum_{t=1}^4 \frac{1}{2} \log(\tau_e) - \frac{1}{2} \tau_e e_{it}^2 \right] + (a_e - 1)\log(\tau_e) - \frac{\tau_e}{b_e}$$

To update the random effect u_i we need,

$$\left[\sum_{t=1}^4 -\mu_{it} + y_{it} \log(\mu_{it}) \right] - \frac{1}{2} \tau_u u_i^2$$

To update the random effect e_{it} we need,

$$[-\mu_{it} + y_{it} \log(\mu_{it})] - \frac{1}{2} \tau_e e_{it}^2$$