1 Data Augmentation

1.1 log of an exponential

The exponential with mean λ has density,

$$p(t) = \frac{exp(-t/\lambda)}{\lambda} dt$$

If we make the transformation $x = log(t/\lambda)$ then $t = \lambda exp(x)$ and $dt = \lambda exp(x)dx$ so the density of x is,

$$p(x) = \frac{exp(-\lambda exp(x)/\lambda)}{\lambda} \lambda exp(x)dx = exp(x - exp(x))dx$$

1.2 Data augmentation for Poisson regression

For a Poisson process with with expected number of events in unit time μ the intervals between successive events, t, follow an exponential distribution with mean $\lambda = 1/\mu$ and $x = log(t/\lambda) = log(\mu t) = log(\mu) + log(t)$, the x will have the density,

$$p(x) = exp(x - exp(x))dx$$

If the Poisson regression model has a linear predictor $log(\mu) = \eta$, then $log(t) + \eta$ will have a density $exp(x - exp(x))$.

Approximating $exp(x - exp(x))$ by a single normal distribution would be very inaccurate. So instead we use a mixture distribution based on five normal distributions $N(m_k, v_k)$ $k = 1...5$ with weights w_k. The probability that component k generates an observed time t is $w_k \phi(log(t) + \eta; m_k, v_k)dt$ where $\phi(x; m, v)$ is the density of a normal distribution with mean m and variance v associated with the observation x. The probability that the observation comes from component k is therefore,

$$\frac{w_k \phi(log(t) + \eta; m_k, v_k)}{\sum_{j=1}^{5} w_j \phi(log(t) + \eta; m_j, v_j)} \qquad k = 1...5$$

We can simulate the component by selecting k with these probabilities.