Gibbs sampler for the Peak Flow model

1 The Posterior

$$\begin{aligned} p(\beta,\tau_1,\tau_2,\underline{\alpha},f|\underline{Y}) &\propto p(\underline{Y}|\beta,\tau_1,\tau_2,\underline{\alpha},f)p(\beta)p(\tau_1)p(\tau_2)p(\underline{\alpha}|f)p(f) \\ &\propto p(\underline{Y}|\beta,\tau_1,\tau_2,\underline{\alpha})p(\beta)p(\tau_1)p(\tau_2)p(\underline{\alpha}|f)p(f) \end{aligned}$$

If we integrate over f we obtain,

$$p(\beta,\tau_1,\tau_2,\underline{\alpha}|\underline{Y}) \propto p(\underline{Y}|\beta,\tau_1,\tau_2,\underline{\alpha})p(\beta)p(\tau_1)p(\tau_2)\int p(\underline{\alpha}|f)p(f)df$$

2 Likelihood

$$p(\underline{Y}|\beta,\tau_1,\tau_2,\underline{\alpha}) \propto \tau_1^{17}exp\left\{-1/2\tau_1\sum_{i=1}^{17}\sum_{k=1}^{2}(Y_{i1k}-\alpha_i)^2\right\}\tau_2^{17}exp\left\{-1/2\tau_2\sum_{i=1}^{17}\sum_{k=1}^{2}(Y_{i2k}-\alpha_i-\beta)^2\right\}$$

3 The Precisions

$$\tau_1 \sim G(a_1,b_1)$$

Posterior proportional to

$$\tau_1^{17}exp\left\{-1/2\tau_1\sum_{i=1}^{17}\sum_{k=1}^{2}(Y_{i1k}-\alpha_i)^2\right\}\tau_1^{a_1-1}exp\left\{-\tau_1/b_1\right\}$$

$$\tau_1^{a_1+17-1}exp\left\{\tau_1\left[1/2\sum_{i=1}^{17}\sum_{k=1}^{2}(Y_{i1k}-\alpha_i)^2+1/b_1\right]\right\}$$

Posterior is gamma $G(a_1+17,\left\{1/2\sum_{i=1}^{17}\sum_{k=1}^{2}(Y_{i1k}-\alpha_i)^2+1/b_1\right\}^{-1})$. Similarly the posterior for τ_2 is gamma,

$$G\left(a_2+17,\left\{1/2\sum_{i=1}^{17}\sum_{k=1}^{2}(Y_{i2k}-\alpha_i-\beta)^2+1/b_2\right\}^{-1}\right)$$

4 The Meter effect

$$\beta \sim N(m,t)$$

Posterior proportional to

$$exp\left\{-1/2\tau_2\sum_{i=1}^{17}\sum_{k=1}^{2}(Y_{i2k}-\alpha_i-\beta)^2\right\}exp\left\{-1/2t(\beta-m)^2\right\}$$

which is proportional to

$$exp\left\{-1/2\left[\beta^2(34\tau_2+t)-2\beta[\tau_2\sum_{i=1}^{17}\sum_{k=1}^{2}(Y_{i2k}-\alpha_i)+tm\right]\right\}$$

So the posterior of β is

$$N\left(\frac{\tau_2\sum_{i=1}^{17}\sum_{k=1}^{2}(Y_{i2k}-\alpha_i)+tm}{34\tau_2+t},34\tau_2+t\right)$$

5 The Subject Means

Cannot simulate f but if we integrate over f we can generate α_i conditional on the other $\alpha's$ and the other parameters using the Chinese restaurant algorithm which is equivalent to sampling from the mixture distribution,

$$\sum_{j=1}^{16}\delta(\alpha_j)/(M+16)+MF(\alpha)/(M+16)$$

The information about α_i in the likelihood has the form,

$$L(\alpha_i)=exp\left\{-1/2\left[\tau_1\sum_{k=1}^{2}(Y_{i1k}-\alpha_i)^2+\tau_2\sum_{k=1}^{2}(Y_{i2k}-\alpha_i-\beta)^2\right]\right\}$$

The mixture therefore becomes,

$$\sum_{j=1}^{16}\delta(\alpha_j)L(\alpha_j)/(M+16)+MF(\alpha)L(\alpha)/(M+16)$$

In the first group of terms $L(\alpha_j)$ alters the probability of selecting α_j, while in the final term L(α) contains both constants that change the probability of choosing this option and terms in α that alter the distribution from which the new value is drawn.
If we choose F = N(A,T) then the final term becomes,

$$\sqrt{\frac{T}{2\pi}}exp\left\{-1/2\left[T(\alpha-A)^2+\tau_1\sum_{k=1}^{2}(Y_{i1k}-\alpha)^2+\tau_2\sum_{k=1}^{2}(Y_{i2k}-\alpha-\beta)^2\right]\right\}$$

If we collect together powers of α in the exponent then coefficient of α^2 will be -1/2 times

$$C_2=T+2\tau_1+2\tau_2$$

The coefficent of α will be -1/2 times,

$$2C_1=2\left[TA+\tau_1\sum_{k=1}^{2}Y_{i1k}+\tau_2\sum_{k=1}^{2}(Y_{i2k}-\beta)\right]$$

and finally the constant will be -1/2 times,

$$C_0 = TA^2 + \tau_1 \sum_{k=1}^{2} Y_{i1k}^2 + \tau_2 \sum_{k=1}^{2} (Y_{i2k} - \beta)^2$$

The expression can therefore be written as

$$\sqrt{\frac{T}{2\pi}} exp(-1/2[C_2(\alpha - C_1/C_2)^2 - C_1^2/C_2 + C_0])$$

which gives,

$$\sqrt{\frac{T}{2\pi}} \sqrt{\frac{2\pi}{C_2}} \sqrt{\frac{C_2}{2\pi}} exp(-1/2[C_2(\alpha - C_1/C_2)^2]) exp(-1/2[-C_1^2/C_2 + C_0])$$

The third and fourth terms define a normal distribution with mean C_1/C_2 and precision C_2 and the remaining terms define the constant that is proportional to the probability,

$$\sqrt{\frac{T}{C_2}} exp(-1/2[C_0 - C_1^2/C_2]) * N(C_1/C_2, C_2)$$

Having found terms proportional to the probabilities we can find the constant of proportionality by using the fact that the 17 probabilities must sum to 1.